**PG – 269**

# II Semester M.Sc. Degree Examination, June/July 2014
## (NS) (2006 Scheme)
## MATHEMATICS
## M-201 : Algebra – II

Time : 3 Hours Max. Marks : 80

***Instructions*** : 1) *Answer **any five** questions choosing atleast **two** from **each** Part.*

2) ***Each** question carry **equal** marks.*

PART – A

1. a) Define an algebraic element in an extension of a field. Let K be an extension of a field F. Prove that an element 'a' of K is algebraic over F if and only if F(a) is a finite extension of F. **6**

   b) Prove that every finite extension K of a field F is algebraic and may be obtained from F by the adjunction of finitely many algebraic elements. **6**

   c) Let $a = \sqrt{2}$, $b = \sqrt[4]{2}$, where R is an extension of Q. Verify that a + b and ab are algebraic of degree atmost (deg a) (deg b). **4**

2. a) Let $f(x) \in F[x]$ be degree $n \geq 1$. Then prove that there is an extension E of F of degree atmost n! in which f(x) has n-roots. **6**

   b) Define splitting field of a polynomial over a field F. Determine the splitting field of $x^3 - 2$ over the field Q. **4**

   c) If P is a prime number, prove that the splitting field over F, the field of rationals, of the polynomial $x^P - 1$ is of degree P – 1. **6**

3. a) If the number $\alpha$ satisfies an irreducible polynomial of degree k over the field of rational numbers and k is not a power of 2, then show that $\alpha$ is not a constructible number. **6**

   b) Prove that a polynomial $f(x) \in F[x]$ has a multiple root if and only if f(x) and $f'(x)$ have a non-trivial common factor. **6**

   c) Prove that any finite extension of a field F of characteristic 0 is a simple extension. **4**

4. State and prove the fundamental theorem of Galois theory. **16**

PART – B

5. a) Let V be finite-dimensional vector space over F, prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0. **6**

b) If V is finite-dimensional vector space over F and if $T \in A(V)$ is right-invertible, then show that T is invertible. **4**

c) If V is a finite-dimensional vector space over F and T, $S \in A(V)$ with S as regular, prove that both T and $STS^{-1}$ have the same minimal polynomial over F. **6**

6. a) Let V be a finite-dimensional vector space over a field F. If $\lambda \in F$ is an eigen value of $T \in A(V)$ and if $f(x) \in F[x]$, prove that $f(\lambda)$ is an eigen value of $f(T) \in A(V)$. **4**

b) If V is n-dimensional vector space over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $\{v_1, v_2, ..., v_n\}$ and the matrix $m_2(T)$ in the basis $\{w_1, w_2, ..., w_n\}$ of V(F), then prove that there is a matrix C in $F_n$ such that $m_2(T) = C\, m_1(T)C^{-1}$. **6**

c) If $\lambda_1, \lambda_2, ..., \lambda_k$ in F are distinct characteristic roots of $T \in A(V)$ and if $v_1, v_2, ..., v_k$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2, ..., \lambda_k$, respectively, then prove that $v_1, v_2, ..., v_k$ are linearly independent. **6**

7. a) Let V be a finite-dimensional vector space over a field F. If $T \in A(V)$ is nilpotent of index r, prove that there exists a vector v in V such that $\{v, T(v), ..., T^{r-1}(v)\}$ is linearly independent over F. **4**

b) If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular. **6**

c) Let $T \in A_F(V)$ has all its distinct characteristic roots $\lambda_1, \lambda_2, ..., \lambda_k$ in F. Then show that there is a basis of V in which the matrix of T is of the form

$$\begin{bmatrix} J_1 & 0 & \cdots & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & J_k \end{bmatrix}, \text{ where } J_i = \begin{bmatrix} B_{i1} & 0 & \cdots & 0 \\ 0 & B_{i2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{ir} \end{bmatrix}_i$$

where $B_{i1}, B_{i2}, ..., B_{ir_i}$ are basic Jordan blocks belonging to $x_i$. **6**

8. a) Let V be a finite-dimensional complex inner product space. If $T \in A(V)$ is such that $\langle T(V); V \rangle = 0$ for each $v \in V$, then prove that T = 0. **4**

b) If T is unitary and if $\lambda$ is a characteristic root of T, then prove that $|\lambda| = 1$. **4**

c) State and prove Sylvester's law of inertia for real quadratic forms. **8**